A puzzle about uniform random variables

Louis Abraham poses two interesting puzzles about uniform random variables on his web page `louis.abraham.github.io`. This note is about the second of these puzzles, which is paraphrased below.

> **Puzzle 2**
>
> Suppose that $X_1, X_2, \ldots$ is a sequence of independent $\mathrm{U}(0,1)$ random variables, and let $Y_k = X_1 + X_2 + \ldots + X_k$. Let U_x be the smallest value of k for which $Y_k > x$ and define $u(x) = \mathrm{E}\,(U_x)$. The basic puzzle is to find $u(1)$.
>
> The author shows that, for $0 \leq x \leq 1$, $u(x) = \exp(x)$, and hence $u(1) = e$.
>
> He goes on to consider the more general problem of finding $u(n)$ for positive integer n and shows that for $n \leq x \leq n+1$, $u(x) = \exp(x)P_n(x)$, where $P_n(.)$ is a polynomial of degree n. He gives a recursive scheme for evaluating $P_n(.)$ and provides convincing numerical evidence that
>
> $$u(n) - 2n \to \frac{2}{3} \qquad \text{as } n \to \infty.$$
>
> The author concludes by asking for a nice proof of this result.

Problems of this type, in which the random variables X_i are independent and identically distributed, are studied in the branch of probability theory known as *renewal theory*. The name comes from a particular area of application that involves items that fail after a random period of time and have to be replaced or renewed. A standard example is light bulbs. The random variable X_i represents the time to failure of the ith bulb and x represents time. The process starts at time zero with a new bulb (in the terminology of renewal theory, it is an *ordinary* renewal process) and U_x is the number of bulbs that have been used by time x, including the bulb that is currently in use. In renewal theory, it is more common to work with the random variable N_x, the number of renewals that have occurred by time x, so that $U_x = N_x + 1$. Note that provided the times to failure are continuous random variables, there is zero probability of a failure occurring precisely at time x.

The function $n(x) = u(x) - 1$ giving the expected number of renewals and is a key quantity in renewal theory, called the *renewal function*. Given this, it is perhaps unsurprising that the form of $u(x)$ when the failure times are independent $\mathrm{U}(0,1)$ variables is known and has in fact been independently rediscovered on several occasions. Nonetheless, it remains an interesting and instructive puzzle. My aim here is to provide a few notes on the problem, aimed particularly at people who are not familiar with renewal theory.

First, I show how an explicit expression for $u(x)$ can be obtained using an approach that is standard in renewal theory. In fact this approach provides the full distribution of U_x, not just its expectation. Unfortunately, the formula for $u(x)$ gives little insight into its asymptotic behaviour. However, applying a general asymptotic result for the expected number of renewals to the particular case of independent uniform times to failure shows that it is indeed the case that

$$u(x) - 2x \to \frac{2}{3} \qquad \text{as } x \to \infty.$$

Although the general asymptotic result is simple to apply, it is not easy to prove. Therefore, I outline a more direct approach due to the distinguished mathematician Harry Furstenberg, which I think comes closer to providing a 'nice' proof.

1 Deriving $u(x)$

The following is a standard approach in renewal theory. The key observation is that

$$\Pr(U_x > k) = \Pr(Y_k \leq x) = F_k(x), \text{ say,}$$

where $F_0(x) = 1$. Then

$$\begin{aligned} \Pr(U_x = k) &= \Pr(U_x > k-1) - \Pr(U_x > k) \\ &= F_{k-1}(x) - F_k(x). \end{aligned} \tag{1}$$

Note that $F_k(x) = 1$ for $k \leq \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x, so that the probabilities will be non-zero only for $k \geq \lfloor x \rfloor + 1$.

From equation (1), and using the fact that $F_k(x) \to 0$ as $k \to \infty$, we have

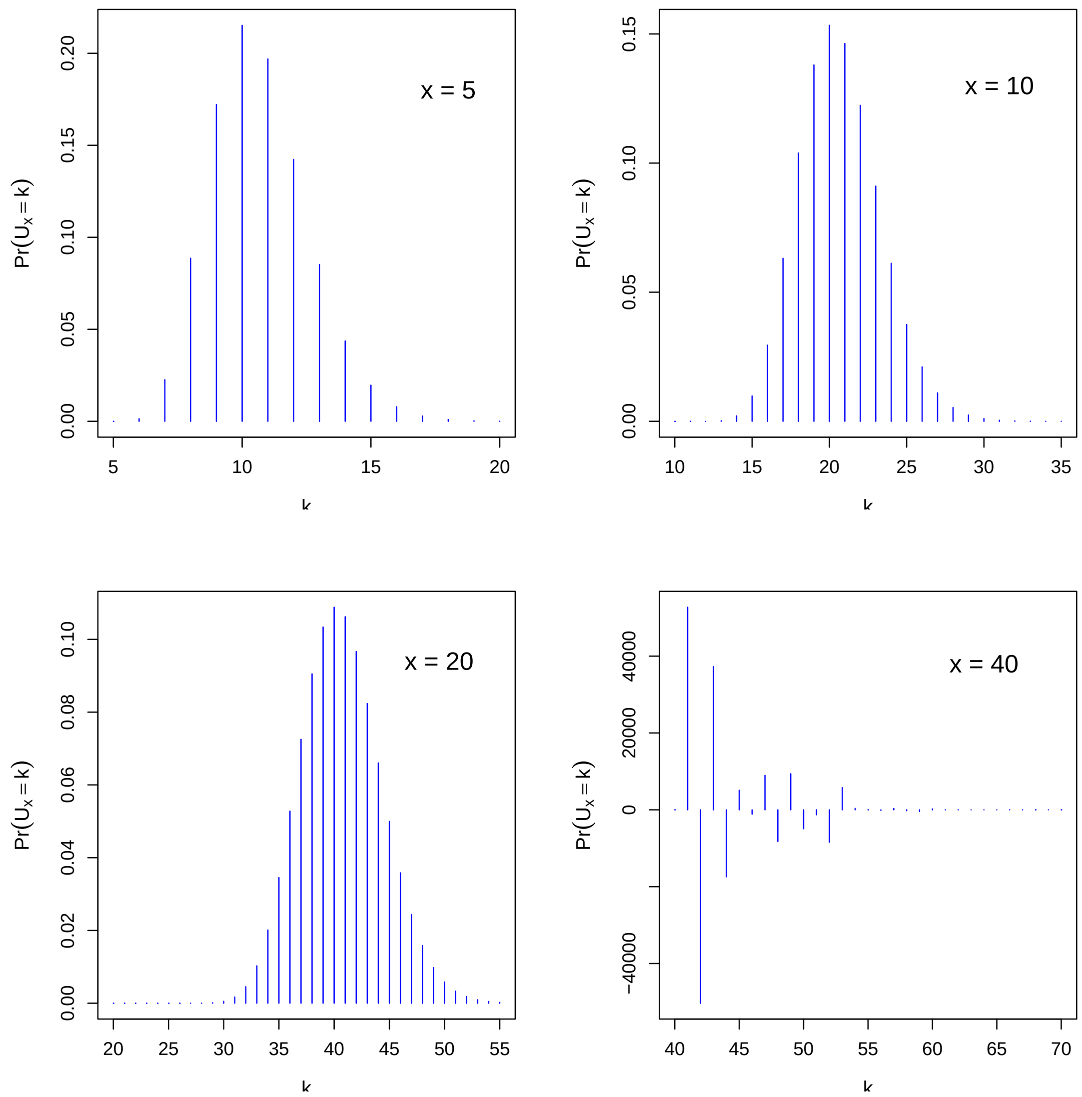
$$\begin{aligned} u(x) &= \sum_{k=1}^{\infty} k \Pr(U_x = k) \\ &= \sum_{k=1}^{\infty} k \left[F_{k-1}(x) - F_k(x) \right] \\ &= \sum_{k=1}^{\infty} F_{k-1}(x) \\ &= 1 + \sum_{k=1}^{\infty} F_k(x). \end{aligned} \tag{2}$$

We can make further progress with this because the distribution of sums of independent $\mathrm{U}(0,1)$ random variables is well studied. For example, it appears in Feller (1971, p.27). Wikipedia refers to it as the Irwin-Hall distribution and gives the following expression for the cumulative distribution function

$$F_k(x) = \frac{1}{k!} \sum_{j=0}^{\lfloor x \rfloor} (-1)^j \binom{k}{j} (x-j)^k. \tag{3}$$

Figure 3 shows the distribution of U_x that results from substituting this expression for

$F_k(x)$ into equation (1) for $x = 5, 10, 20, 40$. A result from renewal theory is that the distribution of U_x is asymptotically normal, and the first three plots do indeed suggest a progressive approach to normality. But for $x = 40$, the calculations have failed completely.



The problem is the well-known instability that often arises when calculating alternating sums with standard double precision arithmetic. Although the problem is clear at $x = 40$, it first arises for much smaller x and can be identified by the occurrence of negative ‘probabilities’. For example, with $x = 14$, the smallest value obtained is -1.2159×10^{-10}.

The simplest way to ensure reliable numerical results is to use a high-precision computing environment.

We can also substitute for $F_k(x)$ in equation (3), giving

$$u(x) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{\lfloor x \rfloor} (-1)^j \binom{k}{j} (x-j)^k.$$

Interchanging the order of summation and simplifying gives

$$u(x) = \sum_{j=0}^{\lfloor x \rfloor} (-1)^j e^{x-j} \frac{(x-j)^j}{j!}. \tag{4}$$

Recall that for $n \leq x \leq n+1$, $u(x) = \exp(x)P_n(x)$, so we have the explicit expression

$$P_n(x) = \sum_{j=0}^{n} (-1)^j e^{-j} \frac{(x-j)^j}{j!}.$$

There is a small subtlety here when $x = n+1$, since the upper limit of summation should then be $n+1$ rather than n. However, it is easy to see that the final term in the summation is then zero.

The argument used to derive $u(x)$ can be modified to obtain expressions for higher-order moments, but we omit the details; a formula for $\text{var}(U_x)$ is given in the next section.

Although it is nice to have the explicit expression (4) for $u(x)$, it is not clear how it can be used to ascertain the asymptotic behaviour of $u(x)$ as $x \to \infty$. For this, we turn to some general asymptotic theory of renewal processes.

2 Limiting form for $u(x)$ as $x \to \infty$

I mentioned earlier that equation (4) has been discovered independently on several occasions. I haven't investigated this systematically, but here are a couple of examples, the second of which leads us to the relevant asymptotic theory.

Suzuki (2004) considers the problem in terms of the hitting time of a random walk and finds the distribution of U_x for $x \in [0, 1]$ and the expectation function $u(x)$ for $x \in [0, 2]$, noting that $u'(x)$ has a discontinuity at $x = 1$, although $u(x)$ itself is continuous there. In a subsequent paper, he gives the distribution of U_x for $x \in [1, 2]$ and the expectation function $u(x)$ for $x \in [2, 3]$, noting that $u'(x)$ is continuous at $x = 2$, but that $u''(x)$ is discontinuous there. He also conjectures the general form of the distribution function for the sum of k independent $U(0, 1)$ random variables and, from this, deduces the correct general expression for $u(x)$.

Russell (1983) studies the problem using an approach similar to that of the previous section, although he obtains the probability generating function of U_x, from which he can easily obtain the variance of $U(x)$ as well as $u(x)$; see Cox (1962) Section 3.2 for a general account of this approach. The variance is

$$\text{var}(U_x) = u(x)\,[2x + 1 - u(x)]\,,$$

implying that $u(x) \le 2x + 1$, since the variance cannot be negative. Russell also obtains the distribution of U_x, although his equation (2) appears to be incorrect.

In a subsequent letter, Jensen (1984) notes that Russell's results are largely rediscoveries of known results in renewal theory. Jensen also gives the result $u(x) = 2x + \frac{2}{3} + o(1)$ as $x \to \infty$, citing Feller (1971, p. 385). The reference is to an exercise in Feller, which starts by asking for a proof of the formula for $u(x)$ and continues

> This formula is frequently rediscovered in queueing theory, but it reveals little about the nature of $u(.)$. The asymptotic formula $0 \le u(x) - 2x \to \frac{2}{3}$ is much more interesting. It is an immediate consequence of (3.1).

Equation (3.1) is part of a general theorem that covers sums of independent and identically distributed random variables. Whilst the proof of this theorem is short, it builds on other theorems.

Smith (1959) studies the cumulants of the number of renewals, N_x. Let $\kappa_n(x)$ denote the nth cumulant of N_x. For $n > 1$, $\kappa_n(x)$ is also the nth cumulant of $U_x = N_x + 1$, whilst for $n = 1$, we have

$$u(x) = \text{E}(U_x) = \text{E}(N_x) + 1 = \kappa_1(x) + 1.$$

Smith shows that

$$\kappa_n(x) = a_n x + b_n + o(1) \qquad \text{as } x \to \infty,$$

where a_n depends on the first n moments of X_i and b_n depends on the first $n+1$ moments. Using Smith's results,

$$u(x) = \frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + o(1) \qquad \text{as } x \to \infty.$$

Smith (1959) gives references to previous proofs of this result, the earliest being Täcklind (1945).

When $X_i \sim \text{U}(0, 1)$, the rth moment is

$$\mu_r = \text{E}\left(X_i^r\right) = \frac{1}{r+1},$$

and the previous result leads to

$$u(x) - 2x \to \frac{2}{3} \qquad \text{as } x \to \infty.$$

For the variance, we have

$$\text{var}(U_x) = \kappa_2(x) = \left(\frac{\mu_2}{\mu_1^3} - \frac{1}{\mu_1}\right) x + \left(\frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{\mu_1^3} - \frac{\mu_2}{2\mu_1^2}\right) + o(1) = \frac{2}{3}x + \frac{2}{9} + o(1) \qquad \text{as } x \to \infty,$$

a result that is also given by Jensen (1984). As a check, this agrees with the result of substituting the limiting form for $u(x)$ into the formula for $\mathrm{var}(U_x)$.

3 A more direct proof

Rather than appealing to general asymptotic results, it is natural to ask whether a simpler proof might result by utilising the fact that the X_i's are independent $\mathrm{U}(0,1)$ variables. The following is an argument of this type due to Furstenberg (1963). I learned of this reference from Spencer (2002).

The starting point is the equation

$$u(x) = 1 + \int_{x-1}^{x} u(t)\,\mathrm{d}t, \tag{5}$$

which appears in the original blogpost. Actually, Furstenberg gives the equation (his equation (1)) as

$$u(x) = F_1(x) + \int_{x-1}^{x} u(t)\,\mathrm{d}t,$$

but this is incorrect for $0 < x < 1$. Furstenberg's equation applies to the renewal function, $n(x)$, not to $u(x)$. However, this is of no great importance since we are interested in the behaviour of $u(x)$ for large x,

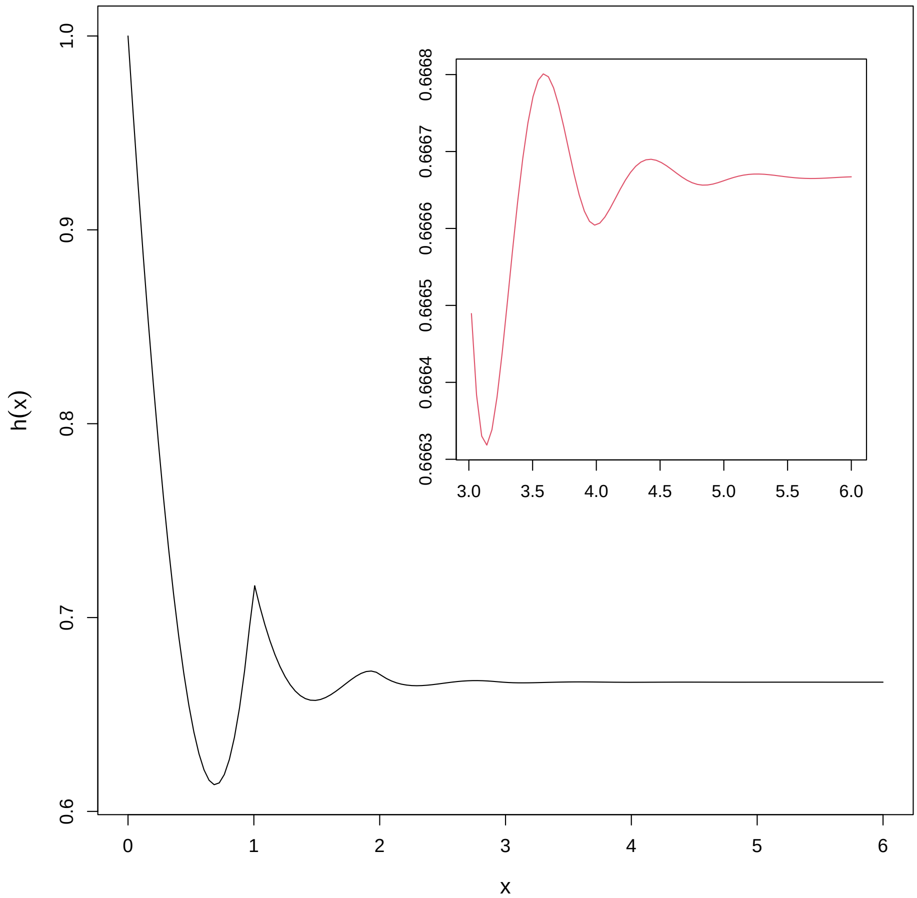
Let $h(x) = u(x) - 2x$. It follows from (5) that

$$\begin{aligned} h(x) &= 1 + \int_{x-1}^{x} [u(t-2t)]\,\mathrm{d}t + \int_{x-1}^{x} 2t\,\mathrm{d}t - 2x \\ &= \int_{x-1}^{x} h(t)\,\mathrm{d}t. \end{aligned}$$

The next step of Furstenberg's argument is to show that $h(x)$ tends to a limit as $x \to \infty$. The proof is terse, understandably since he is writing for professional mathematicians who can be assumed to have a strong grasp of analysis. Sadly, my own grasp is too tenuous to allow me to follow the argument properly, but I reproduce it verbatim here for the benefit of those who can. Note that Furstenberg uses $H(.)$ in place of my $h(.)$.

> Now if
>
> $$H(x) = \int_{x-1}^{x} H(t)\,\mathrm{d}t$$
>
> it is easy to show that $H(x)$ tends to a constant. Since $H(t)$ is clearly bounded, $|H'(x)|$ is bounded and the translations $\{H(x+a)\}$ $(a > 1)$ form an equicontinuous family. If $N = \lim_{x\to\infty} \sup H(x)$, we can find a subsequence of $\{H(x+a)\}$ converging uniformly to a function $\overline{H}(x) = \int_{x-1}^{x} \overline{H}(t)\,\mathrm{d}t$, so that $\overline{H} \equiv N$. But then there must have been unit intervals along which $H(x) \geq N - \epsilon$ for any ϵ. This would give $H(X) \geq N - \epsilon$ for x sufficiently large. It follows that $\lim H(x) = N$.

To see what needs to be proved, below is a plot of $h(x)$ for $x \in [0, 6]$. It shows oscillations that are damping rapidly, so that little detail is visible for $x > 3$ and an amplified view of this part of the plot is shown as an inset. It is apparent that $h(x)$ is not differentiable when $x = 1$, due to the non-differentiability of $u(x)$ at this point, as noted by Suzuki (2004).



The plot certainly supports the idea that $h(x)$ tends to a limit, but in terms of a proof, we need to show that the peaks and the troughs of the oscillations tend to the same limit, or more technically that $\lim_{x\to\infty} \sup H(x) = \lim_{x\to\infty} \inf H(x)$.

If we assume that a limit does indeed exist, the final step is to determine its value, c say. To this end, Furstenberg introduces another function,

$$g(x) = \int_0^1 2th(x+t)\,\mathrm{d}t$$

and claims, without giving details, that $\lim_{x\to\infty} g(x) = c$ and that $g(x)$ is independent of x.

The limit is probably clear, but formally we can write

$$\begin{aligned} g(x) &= \int_0^1 2th(x+t)\,\mathrm{d}t \\ &= \int_0^1 2tc\,\mathrm{d}t + \int_0^1 2t\left[h(x+t)-c\right]\,\mathrm{d}t \\ &= c + \int_0^1 2t\left[h(x+t)-c\right]\,\mathrm{d}t. \end{aligned}$$

Hence

$$\begin{aligned} |g(x)-c| &= \left|\int_0^1 2t\left[h(x+t)-c\right]\,\mathrm{d}t\right| \\ &\leq \int_0^1 2t|\,|h(x+t)-c|\,\mathrm{d}t. \end{aligned}$$

Since $h(x) \to c$ as $x \to \infty$, for any $\epsilon > 0$ there is an x, say x_ϵ such that $|\,|h(x)-c| < \epsilon$ whenever $x > x_\epsilon$. It follows that $|g(x)-c| < \epsilon$ whenever $x > x_\epsilon$ and hence that $g(x) \to c$ as $x \to \infty$.

To prove that $g(x)$ is independent of x, we show that $g'(x) = 0$. To this end, we make the substitution $s = x+t$, which leads to

$$\begin{aligned} g(x) &= \int_x^{x+1} 2(s-x)h(s)\,\mathrm{d}s \\ &= 2\int_x^{x+1} sh(s)\,\mathrm{d}s - 2x\int_x^{x+1} h(s)\,\mathrm{d}s \\ &= 2\int_x^{x+1} sh(s)\,\mathrm{d}s - 2xh(x+1). \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}g(x) &= 2(x+1)h(x+1) - 2xh(x) - 2\frac{\mathrm{d}}{\mathrm{d}x}\left\{xh(x+1)\right\} \\ &= 2x\left[h(x+1)-h(x)\right] + 2h(x+1) - 2h(x+1) - 2x\frac{\mathrm{d}}{\mathrm{d}x}h(x+1) \\ &= 0. \end{aligned}$$

The last line follows from the fact that

$$\frac{\mathrm{d}}{\mathrm{d}x}h(x+1) = \frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x+1} h(t)\mathrm{d}t = h(x+1) - h(x).$$

Sharp-eyed readers might object that $h(x+1)$ is not differentiable at $x = 0$. However, we can use the left derivative instead when $x = 0$.

We can therefore evaluate c as g(0), which gives

$$h(x) \to \int_0^1 2th(t)\,\mathrm{d}t \qquad \text{as } x \to \infty.$$

Substituing for $h(.)$ and recalling that for $0 \le t \le 1$, $u(t) = e^t$, gives

$$u(x) - 2x \to \int_0^1 2t \left[e^t - 2t\right] \mathrm{d}t \qquad \text{as } x \to \infty,$$

and it is easy to show, using integration by parts, that the integral on the right hand side evaluates to 2/3.

4 Additional notes

1. *Excess lifetime*

   Another interesting random variable, is Z_x, the time that elapses between x and the next renewal, that is $Z_x = Y_{U_x} - x$. This goes under various names, including the *excess lifetime*, the *residual lifetime*, the *forward recurrence time* and the *overshoot.* For uniformly distributed X_i, Suzuki (2004, 2005) gives the exact distribution of Z_x for $0 \le x \le 2$. It can also be shown that the limiting distribution as $x \to \infty$ has probability density function $f(z) = 2(1 - z)$ for $0 \le z \le 1$.

   A general result for renewal processes is that

   $$\mathrm{E}(Z_x) = \mu_1 u_x - x, \tag{6}$$

   see for example, Tijms (2003, Lemma 2.1.2). Since $\mathrm{E}(Z_x) \ge 0$, this implies that $u_x \ge x/\mu_1$. When the X_i's are uniformly distributed, (6) leads to the lower bound

   $$u(x) - 2x \ge 0.$$

   It also follows from (6) that

   $$\lim_{x\to\infty} \mathrm{E}(Z_x) = \frac{1}{2} \lim_{x\to\infty} [u_x - 2x] = \frac{1}{3}.$$

   Query 4106814 on Stack Exchange asks if it is possible to prove this result from first principles, without reference to u_x. This would open up a different approach to proving the asymptotic result for $u(x)$, via (6). However, there are no responses to this query.

2. *Laplace transforms*

   Laplace transforms are a useful tool in renewal theory. In particular, the Laplace transform of the renewal function $n(x)$ is

   $$n^*(s) = \frac{f^*(s)}{s[1 - f^*(s)]},$$

   where $f^*(s)$ is the Laplace transform of the probability density function of the lifetime distribution. For uniformly distributed lifetimes, $f^*(s) = (1 - e^{-s})/s$ and we obtain

$$n^*(s) = \frac{1 - e^{-s}}{s(s - 1 + e^{-s})}.$$

The behaviour of the renewal function as $x \to \infty$ can be determined by analysing the behviour of this transform as $s \to 0$. Cox & Miller (1965, p. 345) sketch the general approach, but note that the formal proof is difficult.

3. *Generating function*

   Furstenberg's proof applies for any positive real x. If we are only interested in the case where x is a positive integer n, then an interesting alternative approach is provided in response to query 344713 on Math Overflow. The query doesn't make any specific mention of the renewal problem, but asks about the sequence $P_n(n)$. In the responses, it is shown that $u(n) = e^n P_n(n)$ may be obtained as the coefficient of t^n in the series expansion of the surprisingly simple generating function

$$\frac{t}{e^{t-1} - t}.$$

   It is further shown, using techniques from complex analysis, that

$$u(n) = (2n + 2/3)(1 + O(a^n)),$$

   where $0 < a < 1$.

4. *Renewal Theory*

   Many books on stochastic processes provide introductory material on renewal theory, including Grimmett & Stirzaker (2020, Chapter 10), Ross (2019, Chapter 7) and Tijms (2003, Chapters 2 and 8). Cox (1962) is a book length treatment.

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